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An algorithm for computing a standard form for second-order linear q-difference equations

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Abstract

In this article an algorithm is presented for computing a standard form for second order linear q-difference equations. This standard form is useful for determining the q-difference Galois group and the set of Liouvillian solutions of a given equation. © 1997 Elsevier Science B.V.

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1. Introduction

In this article we will present an algorithm for computing a standard form for second-order q-difference equations over the rational function field $k(z)$, where k denotes a finite algebraic extension of $\mathbb{Q}(q)$ and z is a transcendental variable. This standard form will be defined in Section 2. One can read off fairly easily the q-difference Galois group of a system of q-difference equations if one knows that the system is in standard form. Moreover, if the q-difference Galois group of a second-order q-difference equation is not too big (i.e., does not contain $Sl(2, \mathbb{C})$) then one can compute two linearly independent Liouvillian solutions for this equation. For analogous algorithms for second-order differential equations and second-order difference equations we refer to [3] and [1], respectively.

In Section 2 a standard form for q-difference equations will be defined. Further some notation will be fixed. In Section 3 we will discuss first-order homogeneous linear q-difference equations. The algorithm for second-order homogeneous q-difference equations will be explained in Section 4. Section 5 is devoted to examples.

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2. A short introduction to algebraic aspects of q-difference equations

We will use the following notation:

- $q \in \mathbb{C}$, but q is not a root of unity and $q \neq 0$.
- $q_1, q_2, q_3, \ldots$ is a sequence of complex numbers satisfying $q_1 = q$ and $q_{i+1}^{i+1} = q_i$.
- We define recursively $K_1 = \mathbb{C}(z_1)$ and $K_{i+1} = \mathbb{C}(z_{i+1})$, where $z_{i+1}^{i+1} = z_i$. Instead of K_1 and z_1 we will write K and z.
- $K_\infty = \bigcup_{i=1}^{\infty} K_i$.
- ϕ denotes the $\mathbb{C}$-linear automorphism of K_∞ given by $\phi(z_i) = q_i z_i$. Obviously, ϕ can be restricted to the fields K_i. We will denote these restrictions also by ϕ.

We are interested in the Galois theory of q-difference equations over the field K_∞, because if one works over this field there is a standard form available analogous to the standard form which is defined for ordinary difference equations in [1]. This is not the case if one works over the smaller field $K = \mathbb{C}(z)$. Consider the system of difference equations $(A) : \phi y = Ay$, where $A \in Gl(n, K_\infty)$. (We restrict ourselves to equations with $A \in Gl(n, K_\infty)$ in order to guarantee that we get n independent solutions.)

Definition 1. A ring R together with an automorphism $\phi_R : R \to R$ is called a Picard–Vessiot extension over K_∞ associated with system (A) if

(i) R is a commutative ring, $R \supseteq K_\infty$ and $\phi_R|_{K_\infty} = \phi$.

(ii) The only ϕ_R-invariant ideals are 0 and R.

(iii) There exists a matrix $U \in Gl(n, R)$ such that $\phi_R(U) = AU$. (Such a matrix U is called a fundamental matrix for the system (A).)

(iv) R is minimal with respect to the conditions (i)–(iii) or equivalently if $U = (u_{ij}) \in Gl(n, R)$ is a fundamental matrix for the system (A) then $R = K_\infty[u_{11}, \ldots, u_{nn}, 1/\det(U)]$.

From now on we will denote ϕ_R also by ϕ.

Theorem 2 (Existence and unicity). *For every system of linear q-difference equations (A) with $A \in Gl(n, K_\infty)$ there exist a Picard–Vessiot extension $R \supseteq K_\infty$. This extension is unique up to q-difference K_∞-isomorphism, that is up to K_∞-linear isomorphisms commuting with ϕ. Further we have that $\phi(r) = r$ implies $r \in C$ if $r \in R$.*

Definition 3. The q-difference Galois group $G = DGal(R/K_\infty)$ is the group consisting of all the q-difference K_∞-automorphisms of R.

The vector space of solutions V is the set $\{y \in R^n \mid \phi(y) = Ay\}$. If $U \in GL(n, R)$ is a fundamental matrices for the system (A) then the columns of U form a base for V over $\mathbb{C}$. Suppose that $\sigma \in G = DGal(R/K_\infty)$, then it is obvious that $\sigma(U)$ is also a fundamental matrix for the system (A). Hence $\sigma(U) = UT_\sigma$, where $T_\sigma, T_{\sigma_\infty} \in Gl(n, \mathbb{C})$. So the elements of the q-difference Galois groups act as $\mathbb{C}$-linear maps on the $\mathbb{C}$-linear space of solutions V. But even a stronger statement holds.

Theorem 4 (Algebraicity). *$G = DGal(R/K_\infty)$ is a linear algebraic groups over the field of constants $\mathbb{C}$.*

Theorem 5 (Galois correspondence). (i) *If $a \in R$ then $(\forall \sigma \in G : \sigma(a) = a) \Rightarrow a \in K_\infty$.*

(ii) *If H is an algebraic subgroup of G such that $K_\infty = \{a \in R \mid \forall \sigma \in H : \sigma(a) = a\}$ then $H = G$.*

Theorems 2, 4 and 5 are special cases of theorems for more general difference fields which are proved in ch. 1 of [4].

If G is an algebraic subgroup of $Gl(n, \mathbb{C})$, then we denote by $G(K_\infty)$ the subgroup of $Gl(n, K_\infty)$ which is defined by the same equations. Two systems (A) and (B) corresponding to matrices $A, B \in Gl(n, K_\infty)$ are defined to be equivalent over K_∞ if there exists a $T \in Gl(n, K_\infty)$ such that $B = \phi(T)AT^{-1}$. In this case, if $U \in Gl(n, R)$ is a fundamental matrix of system (A) then TU is a fundamental matrix of system (B) and it is obvious that the solution spaces V_A, V_B of the systems (A) and (B) are equivalent as representation spaces of the q-difference Galois group G. Conversely if the Picard–Vessiot extensions associated to the systems (A) and (B) are q-difference K_∞-isomorphic and the solution spaces V_A and V_B are equivalent as representation spaces of the q-difference Galois group G then (A) and (B) are equivalent systems of q-difference equations.

The algorithms in Section 3 and Section 4 are based on Theorem 6 and the converse of Theorem 7, respectively. These theorems are not valid for general difference fields.

Theorem 6. *Let $G \subseteq Gl(n, \mathbb{C})$ be the q-difference Galois group associated to the system of difference equations (A). Then the following statements hold:*

(i) *G/G^0 is a finite cyclic group.*

(ii) *There exists a $B \in G(K_\infty)$ such that (A) and (B) are equivalent systems.*

It is a conjecture that for every linear algebraic group $G \subseteq Gl(n, \mathbb{C})$ with G/G^0 finite cyclic there exists a system of difference equations which has G as difference Galois group. One can show that this conjecture is true for $n = 1,\ 2$ by giving explicit examples for every possible group.

Theorem 7. *If $A \in G(K_\infty)$ with G an algebraic subgroup of $GL(n, \mathbb{C})$ such that for any proper algebraic subgroup H and for any $T \in G(K_\infty)$ one has that $\phi(T)AT^{-1} \notin H(K_\infty)$ then G is the difference Galois group of the system (A).*

Definition 8. If A satisfies the conditions of Theorem 7 then we say that system (A) is in standard form.

This standard form is in general not unique. But one can read off fairly easily the q-difference Galois group of a system of q-difference equations if one knows already that this system is in standard form. The standard form is also very suitable for finding

Liouvillian solutions. This will be explained in Section 5. In fact, we will present an algorithm which computes for a given equation an equivalent system in standard form.

The proofs of Theorems 6 and 7 are completely analogous to the proofs of Propositions 1.20 and 1.21 in [4] if one applies the next two lemmas.

Lemma 9. *K_∞ is a quasi-algebraically closed (or C_1) field.*

Proof. A field L is called quasi-algebraically closed if it satisfies the condition that if P is a homogeneous polynomial of degree $d \neq 0$ in n variables with coefficients in L whose only zero in L^n is $(0,\dots,0)$, then $d \geq n$. Let P be a homogeneous polynomial of degree $d \neq 0$ in n variables with coefficients in K_∞. Assume that $(0,\dots,0)$ is the only zero of P in K_∞^n. There exists an $m \in \mathbb{Z}_{\geq 1}$ such that the coefficients of P are in K_m. The fields K_i are extensions of transcendence degree 1 of an algebraically closed field. Such fields are well known to be quasi-algebraically closed. Hence $d \geq n$. $\square$

A consequence of Lemma 9 is that if G is a connected group then

$$H^1(Gal(\bar{K}_\infty/K_\infty), G(\bar{K}_\infty)) = 0,$$

where $\bar{K}_\infty$ denotes the algebraic closure of K_∞; see [5].

Lemma 10. *The automorphism ϕ does not extend to any proper finite extension L of K_∞.*

Proof. Suppose L is a finite algebraic extension of K_∞. Then $L = K_\infty(\alpha)$ for a certain $\alpha \in L$. The coefficients of the monic irreducible polynomial for α over K_∞ are in K_m for an $m \in \mathbb{Z}$. Let ρ denote the morphism of algebraic curves $X \to \mathbb{P}(\mathbb{C})$ corresponding to $K_m(\alpha) \supset K_m$. If ϕ extends to $K_m(\alpha)$ then ϕ will permute the finite ramification points of ρ. The automorphism ϕ of $\mathbb{P}(\mathbb{C})$ leaves no finite subset of $\mathbb{P}(\mathbb{C})$ invariant. This implies that ρ is a finite cyclic covering whose only ramification points are 0 and ∞. Hence, $K_m(\alpha) \subset K_\infty$. $\square$

The equations that we look at in the next sections are defined over a small field $k(z) \subset \mathbb{C}(z)$, where k denotes a finite algebraic extension of $\mathbb{Q}(q)$. The algorithm works in the field K_∞, but in reality we will use only a small part of this field. In the main part of the algorithm (Section 4.1) it suffices to work over fields $l(z_2)$, where $l \subset \mathbb{C}$ is a degree 2 extension of k.

3. First order q-difference equations

We consider the first-order homogeneous linear difference equation $(a): \phi y = ay$ with $a \in k(z)^*$, where k is a finite algebraic extension of $\mathbb{Q}(q)$. Any equivalent equation is given by $\phi y = (\phi(f)/f)ay$, where $f \in K_\infty^*$. The q-difference Galois group G of (a) is a finite cyclic group of order n if there is a $f \in K_\infty^*$ such that $(\phi(f)/f)a$ is a primitive nth-root of unity, otherwise the difference Galois group is the multiplicative group $G_m = \mathbb{C}^*$. This statement is a consequence of Theorems 6 and 7.

Suppose $a \in k(z)^*$. Then we can write $a = cz^m(P/Q)$, where $c \in k$, $m \in \mathbb{Z}$ and $P, Q \in k[z]$ are monic polynomials with $\gcd(P,Q) = \gcd(P,z) = \gcd(Q,z) = 1$. Notice that c, m, P, Q are uniquely determined by a.

Definition 11. Suppose that $a = cz^m(P/Q) \in k(z)^*$. Then a is reduced if

(i) For all $n \in \mathbb{Z}$ we have $\gcd(P, \phi^n Q) = 1$.

(ii) If $c = q^r\omega$ where $r \in \mathbb{Q}$ and ω is a root of unity then $r = 0$.

If a is reduced then the equation $(a) : \phi(y) = ay$ is automatically in standard form.

Suppose $a \in K$ is given. We describe an algorithm which produces an element $b \in K$ such that the difference equations (a) and $(b) : \phi(y) = by$ are equivalent and b is reduced.

We assume that $a = cz^m(P/Q)$, where $c \in k$, $m \in \mathbb{Z}$ and $P, Q \in k[z]$ are monic polynomials with $\gcd(P,Q) = \gcd(P,z) = \gcd(Q,z) = 1$. If $c = q^r\omega$ where $r \in \mathbb{Q}$ and ω is a root of unity then we replace c by $\omega = (\phi(z^{-r})/z^{-r})c$.

Write $P = P_1P_2 \cdots P_s$ and $Q = Q_1Q_2 \cdots Q_t$ where the P_i and Q_j are monic and irreducible in $k[z]$. First we take P_1 and we test whether there exist a Q_j and an $n \in \mathbb{Z}$ such that $\phi^n P_1 = Q_j$. If $\phi^n P_1 = Q_j$ then we can go further with $\tilde{a} = a(Q_j/P_1)$. This gives a reduction of the problem with respect to the number of irreducible factors of P and Q, and then we proceed with $\tilde{a}$ instead of a and so forth.

There is also an algorithm which avoids factorization in $k[z]$ if $|q| \neq 1$. We will present this algorithm here. If $|q| \neq 1$ there exists an $R > 0$ such that the zeros of P and Q are in the annulus $1/R \leq |z| \leq R$. We can express R in terms of the coefficients of P and Q. For instance, if $P = P_c z^c + \cdots + P_1 z + P_0$ and $Q = Q_d z^d + \cdots + Q_1 z + Q_0$ then we can take

$$R = 1 + \max\left(\sum_{i=0}^{c-1} \frac{|P_i|}{|P_c|}, \sum_{j=0}^{d-1} \frac{|Q_j|}{|Q_d|}, \sum_{i=1}^{c} \frac{|P_i|}{|P_0|}, \sum_{j=1}^{d} \frac{|Q_j|}{|Q_0|} \right).$$

For $m \in \mathbb{Z}$ with $|q^m| > R^2$ or $|q^m| < 1/R^2$ we have $\gcd(P, \phi^m Q) = 1$. We start with $m = 1$ and compute $g = \gcd(P, \phi(Q))$. If $g \neq 1$ then one can write $a = (g/\phi^{-1}(g))\tilde{P}/\tilde{Q}$ and we can go further with $\tilde{a} = \tilde{P}/\tilde{Q}$ instead of a. This gives a reduction of the problem with respect to the degrees of P and Q. If $g = 1$ then we try $m = -1$ and so forth. The algorithm stops if a is reduced to a constant or if all $m \in \mathbb{Z}$ with $1/R^2 \leq |q^m| \leq R^2$ are checked off.

The above algorithms can easily be modified so that we also find an $f \in K_\infty$ such that $b = (\phi(f)/f)a$ is reduced.

4. Second-order q-difference equations

Consider the second-order linear homogeneous q-difference equation $\phi^2 y + a\phi y + by = 0$, where a, $b \in k(z)$ and $b \neq 0$. As usual, we will identify this equation with the 2×2 system of q-difference equations (A):

$$\phi y = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix} y.$$

Throughout this section we will denote the q-difference Galois group over K_∞ of above system by G.

We present an algorithm which produces a matrix $B \in Gl(2, K_\infty)$ so that the systems (A) and (B) are equivalent and (B) is in standard form. Further a matrix T is computed such that $B = \phi(T)AT^{-1}$.

Lemma 12. *The algebraic subgroups G of $Gl(2,\mathbb{C})$ that can occur as q-difference Galois group over K_∞ are*

(i) *Any reducible subgroup of $Gl(2,\mathbb{C})$ with G/G^0 finite cyclic.*
(ii) *Any infinite imprimitive subgroup of $Gl(2,\mathbb{C})$ with G/G^0 finite cyclic.*
(iii) *Any subgroup containing the group $Sl(2,\mathbb{C})$.*

Proof. This is a consequence of Theorem 6. and the well-known classification of the algebraic subgroups of $Gl(2,\mathbb{C})$. $\square$

The algorithm is arranged in the following manner. Given a second-order q-difference equation we test which of the three cases above holds. After that we compute an appropriate equivalent system in standard form.

4.1. The Riccati equation

To a second-order linear q-difference equation $\phi^2 y + a\phi y + by = 0$, where $a, b \in k(z)$ and $b \neq 0$ we can associate a first-order non-linear q-difference equation $(\dagger) : u\phi(u) + au + b = 0$. This equation is called the Riccati equation. If u is a solution of the Riccati equation then the q-difference operator $\phi^2 + a\phi + b$ factors as $(\phi - b/u)(\phi - u)$. A solution in K_∞ of the Riccati equation corresponds to a line of the solution space that is fixed by the q-difference Galois group G. Therefore we have the following theorem; see also [1].

Theorem 13. *The following statements hold:*

(i) *If the Riccati equation has no solutions $u \in K_\infty^*$ then G is irreducible.*
(ii) *If the Riccati equation has exactly one solution $u \in K_\infty^*$ then G is reducible but not completely reducible.*

(iii) *If the Riccati equation has exactly two solutions* $u_1, u_2 \in K_\infty^*$, *then* G *is completely reducible but* G *is not an algebraic subgroup of* $\{c \cdot Id \mid c \in \mathbb{C}^*\}$.

(iv) *If the Riccati equation has more than two solutions in* K_∞^* *then the Riccati equation has infinitely many solutions and* G *is an algebraic subgroup of* $\{c \cdot Id \mid c \in \mathbb{C}^*\}$.

In this section we will describe an algorithm which computes all solutions $u \in K_\infty^*$ of the Riccati equation. The algorithm consists of a few steps. In the first three steps we will compute all solutions $u \in K$ of the Riccati equation. In the first step of the algorithm we will compute the first term of all possible local formal solutions at 0 and ∞. In the next two steps we try to glue these local formal solutions to a global solution. In the fourth step we will prove that if there exists a solution $u \in K_\infty$ then there is also a solution $u \in K_2$. After that we will describe how to find the solutions $u \in K_2$ if we did not find any solution $u \in K$.

(1) Let ϕ be the automorphism of $\mathbb{C}((z))$ given by $\phi(z) = qz$. Then ϕ coincides with our former ϕ on $K = \mathbb{C}(z)$. We define the discrete valuation $v_0 : \mathbb{C}((z)) \to \mathbb{Z}$ by $v_0(\sum_{i=-\infty}^{\infty} c_i z^i) = \min\{i \mid c_i \neq 0\}$. In particular $v_0(0) = \infty$. Elementary properties of v_0 are $v_0(u_1 u_2) = v_0(u_1) + v_0(u_2)$ and $v_0(u_1 + u_2) \geq \min(v_0(u_1), v_0(u_2))$. Hence if $v_0(u_1) \neq v_0(u_2)$ then $v_0(u_1 + u_2) = \min(v_0(u_1), v_0(u_2))$. Another property of v_0 is that $v_0(u) = v_0(\phi u)$. Now we consider a and b as Laurent series in z.

We distinguish two cases.

(a) $2v_0(a) < v_0(b)$. In this case there are two solutions $u, \tilde{u} \in k((z))$ of the Riccati equation. The first solution u satisfies $v_0(au) = v_0(u\phi(u)) < v_0(b)$ and $v_0(u) = v_0(a)$ and the other solution $\tilde{u}$ satisfies $v_0(a\tilde{u}) = v_0(b) < v_0(\tilde{u}\phi(\tilde{u}))$ and $v_0(\tilde{u}) = v_0(b) - v_0(a)$. Suppose $v_0(a) = l$ and $v_0(b) = k$. Then we will write $a = \sum_{i=l}^{\infty} a_i z^i$ and $b = \sum_{i=k}^{\infty} b_i z^i$. We can compute successively the coefficients of u and $\tilde{u}$ by simply solving linear equations with coefficients in k. Hence, no field extension of k is needed. We find $u = -(a_l/q^l)z^l + O(z^{l+1})$, where $O(z^{l+1})$ stands for the higher-order terms and $\tilde{u} = -(b_k/a_l)z^{k-l} + O(z^{k-l+1})$.

(b) $v_0(b) \leq 2v_0(a)$. If $u \in \mathbb{C}((z))$ satisfies the Riccati equation then we have $v_0(b) = v_0(u\phi(u)) \leq v_0(au)$. Hence, if $v_0(b)$ is odd there is obviously no solution of the Riccati equation in the field $\mathbb{C}((z))$. Suppose now $v_0(b)$ is even. Let $n = \frac{1}{2}v_0(b)$. Write $b = \sum_{i=2n}^{\infty} b_i z^i$ and $a = \sum_{i=n}^{\infty} a_i z^i$. (Note that it is possible that $a_n = 0$.) Write $u = \sum_{i=n}^{\infty} b_i z^i$. Then we try to solve the Riccati equation (†): $u\phi(u) + au + b = 0$. The $(2n)$th-coefficient of this expression gives us the equation $q^n u_n^2 + a_n u_n + b_{2n} = 0$. Let $D_0 = a_n^2 - 4b_{2n}q^n$. If $D_0 \neq 0$ then we find two solutions $u_n = ((-a_n \pm \sqrt{D_0})/2q^n) \in k(\sqrt{D_0})$ of the quadratic equation. So $u = ((-a_n \pm \sqrt{D_0})/2q^n)z^n + O(z^{n+1})$. If $D_0 = 0$ then we find one solution $u = (-a_n/2q^n)z^n + O(z^{n+1})$.

Let S_0 be the set of all possibilities for the first term of u expressed as Laurent series in z. The coefficient of this term lies in the field $k(\sqrt{D_0})$ with $[k(\sqrt{D_0}) : k] \leq 2$. We discovered that $\#S_0 \leq 2$, even if the Riccati equation has infinitely many solutions $u \in K_\infty$. In the same way we can determine the set S_∞ of all possibilities for the first term of u expressed as Laurent series in $1/z$ with a coefficient in at worst a quadratic field extension $k(\sqrt{D_\infty})$. We also have $\#S_\infty \leq 2$. For future use we will denote $\tilde{k} = k(\sqrt{D_0}, \sqrt{D_\infty})$.

(2) In this part of the algorithm we will determine the solutions $u \in \tilde{k}(z)$ of the Riccati equation. We rewrite the Riccati equation as $Fu\phi(u) + Gu + H = 0$, where $F, G, H \in k[z]$ and $\gcd(F, G, H) = 1$. Write $u = cz^m(P/T)$ with $c \in \tilde{k}$, $m \in \mathbb{Z}$, $P, T \in \tilde{k}[z]$, where P and T are monic polynomials and $\gcd(P,T) = \gcd(P,z) = \gcd(T,z) = 1$. Let R denote the greatest monic divisor of T such that $\phi(R)$ divides P. Then one can write $u = cz^m(\phi(R)p/Rt)$, where p, t are monic polynomials, $\gcd(p, \phi(t)) = 1$ and $\gcd(\phi(R)p, Rt) = 1$. Further $\gcd(R,z) = \gcd(p,z) = \gcd(t,z) = 1$. The Riccati equation reads now

$$(\ddagger) \qquad c^2 q^m z^{2m} \phi^2(R)\phi(p)pF + cz^m\phi(R)\phi(t)pG + R\phi(t)tH = 0.$$

Hence, p is a monic divisor of H with $v_0(p) = 0$ and t is a monic divisor of $\phi^{-1}(F)$ with $v_0(t) = 0$. We define the sets $S_p = \{p \in \tilde{k}[z] \mid p|H,\ v_0(p) = 0 \text{ and } p \text{ monic}\}$ and $S_t = \{t \in \tilde{k}[z] \mid t|\phi^{-1}(F),\ v_0(t) = 0 \text{ and } t \text{ monic}\}$.

Writing $\phi(R)/R$ as a Laurent series in z we get $\phi(R)/R = 1 + \mathrm{O}(z)$, where $\mathrm{O}(z)$ stands for the higher-order terms. $\phi(R)/R$ must be equal to $ut/cz^m p$. Hence, for every possible combination $u_0 \in S_0$, $p \in S_p$ and $t \in S_t$ we compute a $c \in \tilde{k}$ and a $m \in \mathbb{Z}$ from the equation $u_0 t/cz^m p = 1 + \mathrm{O}(z)$. Let e be the degree of R. Writing $\phi(R)/R$ as a Laurent series in $1/z$ we get $\phi(R)/R = q^e + \mathrm{O}(1/z)$. So we must have $v_\infty(u) - \deg_z(t) + m + \deg_z(p) = 0$. Hence, if there are no terms $u_\infty \in S_\infty$ with $v_\infty(u_\infty) = \deg_z(t) - \deg_z(p) - m$ then there is no solution $u \in K$. For all $u_\infty \in S_\infty$ satisfying $v_\infty(u_\infty) = \deg_z(t) - \deg_z(p) - m$ we compute a $d \in \tilde{k}$ from $u_\infty t/cz^m p = d + \mathrm{O}(1/z)$. If $d = q^e$ with $e \in \mathbb{Z}_{\geq 0}$ then we write $R = z^e + r_{e-1} + \cdots + r_1 z + r_0$ with indeterminates $r_0, r_1, \ldots, r_{e-1}$. After substituting this R in ($\ddagger$) we are left with a set of equations in the r_i. In this way we will find all possible solutions $u \in \tilde{k}(z)$ of the Riccati equation.

(3) In step 2 we have determined all solutions $u \in \tilde{k}(z)$ of the Riccati equation. Now we have to face the problem whether it is possible that there exist solutions $u \in \bar{K}$ if there are no solutions $u \in \tilde{k}(z)$. A priori this is possible because $\phi^{-1}F$ and G need not split in linear factors in $\tilde{k}[z]$. But it is also clear from the construction in step 2 that if the Riccati equation has a solution in $\bar{K}$, there is also a solution in $l(z)$, where $l \supseteq \tilde{k}$ is the smallest field which contains the splitting fields of $\phi^{-1}F$ and G. But we have even a stronger rationality result.

Theorem 14. *If the Riccati equation has a solution in $\bar{K}$ then there is also a solution in a field $\check{k}(z)$ with $[\check{k} : k] \leq 2$.*

Proof. This proof is completely analogous to the proof of Theorem 4.3 in [1]. □

So we want to determine all fields $\check{k} \subset l$ with $[\check{k} : k] \leq 2$. If q is algebraic then $\check{k}$ must be of the form $k(\sqrt{s})$, where s is an algebraic integer in l. Recall that l is the splitting field of the polynomial FG. After a suitable linear substitution $z \mapsto z/n$, where $n \in \mathbb{Z}_{\geq 1}$ we can assume that FG is monic and that the coefficients of FG are algebraic integers. s must be a divisor of the discriminant of this polynomial. This gives us a finite set of possible quadratic extensions $k(\sqrt{s})$ of k in l. If q is transcendental then $\check{k}$ must be of the form $k(\sqrt{s})$ where s is a polynomial in q_i for a certain i with algebraic integer coefficients. After a suitable linear substitution we can assume that FG is monic and the coefficients of FG are polynomials in q_i (choose i minimal) with algebraic integer coefficients. Now s must be a divisor of the discriminant of the polynomial FG which is a polynomial in q_i with algebraic integer coefficients. Again this gives us a finite set of possible quadratic extensions $k(\sqrt{s})$ of k in l.

For each possible quadratic extension $\check{k}$ we determine the sets $S_p = \{p \in \check{k}[z] \mid p|H,$ $v_0(p) = 0$ and p monic$\}$ and $S_t = \{t \in \check{k}[z] \mid t|\phi^{-1}(F),\ v_0(t) = 0$ and t monic$\}$. After that we proceed as in 2.

(4) In steps 1–3 we have determined all solutions $u \in K = \mathbb{C}(z)$ of the Riccati equation. If we did not find any solution $u \in K$ then we try to find solutions $u \in K_\infty \backslash K$. We need the following theorem.

Theorem 15. *If the Riccati equation has a solution in K_∞^* then there is also a solution in K_2^*.*

Proof. Let U be the set of solutions of the Riccati equation in K_∞^*. The group $Gal(K_\infty^*/K)$ acts on the set U. If $\#U = 1$, say $U = \{u\}$, then u is invariant under $Gal(K_\infty^*/K)$. Hence $u \in K^*$. If $\#U = 2$, then the kernel of the map $Gal(K_\infty^*/K) \to Aut(U)$ is a subgroup of index ≤ 2. Hence u is an element of K_2, because K_2 is the only quadratic field extension of K in K_∞. Suppose that $\#U = \infty$. According to Theorems 13 and 6 there exists a $T \in Gl(2, K_\infty)$ such that

$$\phi(T)\begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix} T^{-1} = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}$$

for some $u \in K_\infty$. There is a $l \in \mathbb{Z}$ such that $T \in Gl(2, K_l)$ and $u \in K_l$. Suppose $\sigma \in Gal(K_l/K)$ then

$$\phi(\sigma(T))\begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix} (\sigma(T))^{-1} = \begin{pmatrix} \sigma(u) & 0 \\ 0 & \sigma(u) \end{pmatrix}.$$

Hence, there is a $g_\sigma \in K_l$ such that $\sigma(u) = (\phi(g_\sigma)/g_\sigma)u$. This g_σ is uniquely determined if we require that the numerator and denominator of g_σ are monic. The map $\sigma \mapsto g_\sigma$ is a 1-cocycle. Hilbert 90 states that this 1-cocycle is trivial. See [6]. We can replace u by $\tilde{u} = (\phi(h)/h)u$ for a certain $h \in K_l^*$ such that $\tilde{u}$ is invariant under $Gal(K_l/K)$. Hence $\tilde{u} \in K$. Let $T_1 = hT$. Then

$$\phi(T_1)\begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix} T_1^{-1} = \begin{pmatrix} \tilde{u} & 0 \\ 0 & \tilde{u} \end{pmatrix}.$$

We have $C^{-1}T_1 \in Gl(2, K_\infty)$ satisfies

$$\phi(C^{-1}T_1)\begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix}(C^{-1}T_1)^{-1} = \begin{pmatrix} \tilde{u} & 0 \\ 0 & \tilde{u} \end{pmatrix}$$

if and only if $\phi(C) = C$, that is $C \in Gl(2, \mathbb{C})$. For $\sigma \in Gal(K_l/K)$ one has therefore $\sigma(T_1) = C_\sigma^{-1}T_1$ for some $C_\sigma \in Gl(2, \mathbb{C})$. The map $\sigma \mapsto C_\sigma$ is a homomorphism $Gal(K_l/K) \to Gl(2, \mathbb{C})$. We have $Gal(K_l/K)$ is a finite cyclic group. So there exists an M in $Gl(2, \mathbb{C})$ such that $M^{-1}C_\sigma M$ is a diagonal matrix for all $\sigma \in Gal(K_l/K)$. Let

$$T_2 = M^{-1}T_1 = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}.$$

Then $-t_{21}/t_{22}$ and $-t_{11}/t_{12}$ satisfy the Riccati equation. Moreover, $-t_{21}/t_{22}$ and $-t_{11}/t_{12}$ are fixed by $Gal(K_l/K)$ and therefore $-t_{22}/t_{21}$, $-t_{11}/t_{12} \in K$. $\square$

Now we consider a, b as elements of $k(z_2)$. We apply steps 1–3 of the algorithm with z_2 and q_2 in the role of z and q respectively. In this way we will find all solutions $u \in K_2$ of the Riccati equation. This finishes the algorithm.

4.2. G is reducible

If the Riccati equation has a solution in K_∞, then the q-difference Galois group G is reducible. According to Theorem 6 we can compute an equivalent system (A): $\phi y = Ay$, where A has the form $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$. Moreover we can assume that $b = 0$ if the Riccati equation has two or infinitely many solutions. We denote by U the subgroup of $Gl(2, \mathbb{C})$ consisting of upper triangular matrices and by D the subgroup of diagonal matrices.

Lemma 16. *The reducible subgroups $G \subset Gl(2, \mathbb{C})$ with G/G^0 finite cyclic are*
(i) *The zero-dimensional groups*

$$D_{k,l,e} = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \in D \;\middle|\; \alpha^k = 1, \delta^l = 1, \alpha^{eg_\alpha}\delta^{g_\delta} = 1 \right\}$$

and the one-dimensional groups

$$U_{k,l,e} = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \in U \;\middle|\; \alpha^k = 1, \delta^l = 1, \alpha^{eg_\alpha}\delta^{g_\delta} = 1 \right\}$$

for $l, k, e \in \mathbb{Z}_{\geq 1}$ with $\gcd(k, l, e) = 1$, *where $g_\alpha = (k/\gcd(k, l))$ and $g_\delta = (l/\gcd(k, l))$.*

(ii) *The one-dimensional groups*

$$D_{m,n} = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \in D \;\middle|\; \alpha^n \delta^m = 1 \right\}$$

and the two-dimensional groups

$$U_{m,n} = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \in U \;\middle|\; \alpha^n \delta^m = 1 \right\}$$

for $m, n \in \mathbb{Z}$.

(iii) *The two-dimensional group* D *and the three-dimensional group* U.

Proof. See [1]. □

The algorithm continues as follows. Suppose the Riccati equation has exactly one solution in K_∞. Then the difference Galois group G is reducible but not completely reducible. If G is a proper subgroup of U, then according to Theorem 7 there must be a matrix

$$T = \begin{pmatrix} f & g \\ 0 & h \end{pmatrix} \in U(K_\infty)$$

such that

$$B = \phi(T) A T^{-1} = \begin{pmatrix} a\dfrac{\phi(f)}{f} & * \\ 0 & d\dfrac{\phi(h)}{h} \end{pmatrix} \in G_\infty(K_\infty).$$

First we want to determine whether the q-difference Galois group G is one of the one-dimensional groups described in part 1 of Lemma 16. We apply the algorithm for first order q-difference equations to a and d. This yields a new equivalent system

$$\tilde{A} = \begin{pmatrix} \tilde{a} & * \\ 0 & \tilde{d} \end{pmatrix},$$

where $\tilde{a}$ and $\tilde{d}$ are reduced.

Suppose first that $\tilde{a}$ and $\tilde{d}$ are constant. Then the system $\tilde{A}$ is already in standard form if $\tilde{a}$ or $\tilde{d}$ is a root of unity. If both $\tilde{a}$ and $\tilde{d}$ are roots of unity, then the difference Galois group is one of the groups $U_{k,l,e}$. The q-difference Galois group G is equal to $U_{m,0}$, if $\tilde{a}$ is a primitive mth root of unity and $\tilde{d}$ is not a root a unity. And the q-difference Galois group G is equal to $U_{0,n}$, if $\tilde{d}$ is a primitive nth root of unity and $\tilde{a}$ is not a root a unity. If $\tilde{a}$ and $\tilde{d}$ are constant but not roots of unity then if there exist $n, m \in \mathbb{Z}$ such that $\tilde{a}^m \tilde{d}^n = q^r$ with $r \in \mathbb{Q}$ the q-difference Galois group G is one of the groups described in part 2 of Lemma 16. Moreover, suppose that m is minimal and positive. In that case G is equal to the group $U_{m,n}$. If $r = 0$ then the system is already in standard form. Suppose now that $r \neq 0$. It is obvious that there exists an $f \in K_\infty$ such that $(\phi(f)/f)^n = q^{-r}$. Let

$$T = \begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $B = \phi(T)\tilde{A}T^{-1}$ is in standard form.

Suppose now that $\tilde{a}$ and $\tilde{d}$ are not constant. Then the q-difference Galois group is not a group described in part 1 of Lemma 16.

If $\tilde{a}$ is constant and reduced and $\tilde{d}$ is not constant and reduced then the system $(\tilde{A})$ is already in standard form. The q-difference Galois group G is the group $U_{m,0}$, if $\tilde{a}$ is an mth primitive root of unity. Otherwise the q-difference Galois group G is the group U.

If $\tilde{d}$ is constant and reduced and $\tilde{a}$ is not constant and reduced then the system $(\tilde{A})$ is also already in standard form. The q-difference Galois group G is the group $U_{0,n}$, if $\tilde{d}$ is an nth primitive root of unity. Otherwise the q-difference Galois group G is the group U.

Suppose now that both $\tilde{a}$ and $\tilde{d}$ are reduced but not constant. The q-difference Galois group is one of the groups $U_{m,n}$ if and only if there exists $r, s \in \mathbb{Z}\backslash\{0\}$ with $\gcd(r,s) = 1$ and an $f \in K_\infty$ such that $\tilde{a}^r\tilde{d}^s(\phi(f)/f)$ is a root of unity. Without loss of generality, we can assume that $r > 0$. Let N_a and N_d denote the degree of the numerator of $\tilde{a}$ and $\tilde{d}$, respectively, and let D_a and D_d denote the degree of the denominator of $\tilde{a}$ and $\tilde{d}$, respectively.

Suppose first that there exist $r, s \in \mathbb{Z}_{\geq 1}$ with $\gcd(r,s) = 1$ and an $f \in K_\infty$ such that $\tilde{a}^r\tilde{d}^s(\phi(f)/f)$ is constant. Then we must have $rD_a = sN_d$ and $rN_a = sD_d$, because $\tilde{a}$ and $\tilde{d}$ are reduced.

If $rD_a = sN_d$ and $rN_a = sD_d$ hold then we apply the algorithm for first-order q-difference equations to $\tilde{a}^r\tilde{d}^s$ to investigate whether there exist an $f \in K_\infty$ such that $\tilde{a}^r\tilde{d}^s(\phi(f)/f)$ is constant and reduced. If there exist such an f then we let

$$T = \begin{pmatrix} f^v & 0 \\ 0 & f^w \end{pmatrix},$$

where v, w are chosen such that $vr + ws = 1$. Then $B = \phi(T)\tilde{A}T^{-1}$ is in standard form.

Suppose now that there exists an $r \in \mathbb{Z}_{\geq 1}$ and an $s \in \mathbb{Z}_{\leq -1}$ with $\gcd(r,s) = 1$ and an $f \in K_\infty$ such that $\tilde{a}^r\tilde{d}^s(\phi(f)/f)$ is constant and reduced. Then we must have $rN_a = -sN_d$ and $rD_a = -sD_d$, because $\tilde{a}$ and $\tilde{d}$ are reduced.

If $rN_a = -sN_d$ and $rD_a = -sD_d$ hold then we apply the algorithm for first-order q-difference equations to $\tilde{a}^r\tilde{d}^s$ to investigate whether there exist an $f \in K_\infty$ such that $\tilde{a}^r\tilde{d}^s(\phi(f)/f)$ is constant and reduced. If there exist such an f then we let

$$T = \begin{pmatrix} f^v & 0 \\ 0 & f^w \end{pmatrix},$$

where v, w are chosen such that $vr + ws = 1$. Then $B = \phi(T)\tilde{A}T^{-1}$ is in standard form.

If none of the two cases above hold then the system $(\tilde{A})$ is already in standard form and the q-difference Galois group is the upper triangular group U.

We will not discuss the completely reducible case because this case is analogous to the reducible but not completely reducible case.

4.3. *G is imprimitive*

If we did not find a solution $u \in K_\infty^*$ for the Riccati equation then the q-difference Galois group G is irreducible. A group G is called imprimitive if

$$G \subseteq F = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \middle| \alpha\delta \neq 0 \right\} \cup \left\{ \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} \middle| \beta\gamma \neq 0 \right\}.$$

In this subsection we determine whether G is imprimitive if we know already that G is irreducible.

Lemma 17. *Suppose that the q-difference Galois group G of the difference equation (#): $\phi^2 y + a\phi y + by = 0$ is irreducible. Then the following statements are equivalent.*

(i) *G is an imprimitive group.*

(ii) *There exists an $r \in K_\infty^*$ such that the equations (#) and $\phi^2 y + ry = 0$ are equivalent.*

Proof. See [1]. □

Suppose now that the equations (#): $\phi^2 y + a\phi y + by = 0$ with $a \neq 0$ and $\phi^2 y + ry = 0$ are equivalent. Then there exist $c, d \in K_\infty^*$ such that y is a solution of (#) if and only if $cy + d\phi y$ is a solution of $\phi^2 y + ry = 0$. In this case $(c/d)y + \phi y$ satisfies the equation $\phi^2 y + (d/\phi^2(d))ry = 0$.

Theorem 18. *Suppose that the q-difference Galois group G of the equation (#): $\phi^2 y + a\phi y + by = 0$, $a \neq 0$ is irreducible. Then G is imprimitive if and only if there exists a solution $E \in K_\infty^*$ of the Riccati equation*

$$(\#\#) \qquad \phi^2(E)E + \left(\phi^2\left(\frac{b}{a}\right) - \phi(a) + \frac{\phi(b)}{a} \right) E + \frac{\phi(b)b}{a^2} = 0.$$

And if $E \in K_\infty^*$ is a solution of this Riccati equation then y satisfies the equation (#) if and only if $Dy + \phi y$ satisfies the equation $\phi^2 y + ry = 0$, where $D = E + b/a$ and $r = -a\phi(a) + \phi(b) + a\phi^2(D)$.

Proof. See [1]. □

Remark 19. We can find the rational solutions of equation (##) by applying the algorithm in Section 4.1 with q replaced by q^2.

Lemma 20. *The imprimitive subgroups $G \subset Gl(2,\mathbb{C})$ with G/G^0 finite cyclic are*

$$F = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \middle| \alpha\delta \neq 0 \right\} \cup \left\{ \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} \middle| \beta\gamma \neq 0 \right\}.$$

$$H_n^- = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \middle| (\alpha\delta)^n = 1 \right\} \cup \left\{ \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} \middle| (\beta\gamma)^n = -1 \right\} \text{ for } n \in \mathbb{Z}_{\geq 1}.$$

$$H_n^+ = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \middle| (\alpha\delta)^n = 1 \right\} \cup \left\{ \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} \middle| (\beta\gamma)^n = 1 \right\} \text{ for } n \text{ odd}.$$

Proof. See [1]. □

Consider the system

$$\phi(y) = \begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix} y.$$

We can apply the algorithm for first order q-difference equations to cd. Then we will find an $f \in K_\infty^*$ such that $cd(\phi(f)/f)$ is reduced. Let

$$T = \begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$B = \phi(T) \begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix} T^{-1}$$

is in standard form.

4.4. *G contains $Sl(2,\mathbb{C})$*

Consider the system

$$\phi(y) = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix} y.$$

If the q-difference Galois group G of this system is neither reducible nor imprimitive, then $G = Gl(2,\mathbb{C})$ or

$$G = Sl(2,\mathbb{C})_n = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \middle| (\alpha\delta - \beta\gamma)^n = 1 \right\}.$$

We apply the algorithm for first order q-difference equations to

$$b = \det \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix}$$

and find an $f \in K_\infty^*$ such that $(\phi(f)/f)b$ is reduced. Let

$$T = \begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$B = \phi(T) \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix} T^{-1}$$

is in standard form. This finishes the algorithm.

5. Examples

In this section sequences spaces are used in order to make the Picard–Vessiot rings associated to a system of q-difference equations more concrete. The explicit solutions which can be computed are certain sequences which will be called Liouvillian. The second-order q-difference equations which will be treated in Section 5.2 are the so-called hypergeometric q-difference equations.

5.1. Sequences spaces

Let F be any field. Consider the set of sequences $F^{\mathbb{N}}$. This set forms a ring by using coordinate-wise addition and multiplication. Let J be the ideal of sequences with at most finitely many non-zero terms. Then we define $\mathscr{S}_F = F^{\mathbb{N}}/J$. The map $\rho : \mathscr{S}_F \to \mathscr{S}_F$ given by $\rho((a_1, a_2, a_3, \ldots)) = (a_2, a_3, \ldots)$ is well defined on equivalence classes. Moreover, ρ is an automorphism of the ring $\mathscr{S}_F$. For simplicity we will identify an element $a = (a_1, a_2, a_3, \ldots)$ with its equivalence class.

Consider the fields $K_i = \mathbb{C}(z_i)$. We can embed the fields K_i in $\mathscr{S}_{\mathbb{C}}$ by mapping a rational function $f(z_i)$ to the sequence $s_f = (s_1, s_2, \ldots)$, where $s_n = f(q_i^n)$ for all but finitely many $n \in \mathbb{N}$. We will denote the homomorphisms $f \mapsto s_f$ by ψ. We have $\psi(z_i) = \psi(z_{i+1})^{i+1}$. Hence, the embeddings of the fields K_i induce an embedding of the field K_∞. It is obvious that $\psi\phi = \rho\psi$. The map $\psi : K_\infty \to \mathscr{S}_{\mathbb{C}}$ is an injective homomorphism because K_∞ is a field and $\psi(1) = 1$. Therefore, from now on we will identify an element $f \in K_\infty$ with its image s_f in $\mathscr{S}_{\mathbb{C}}$.

Theorem 21. *Consider the system of q-difference equations $(A) : \phi(y) = Ay$, where $A \in Gl(n, K_\infty)$. There exist a q-difference subring $R \subset \mathscr{S}_{\mathbb{C}}$ such that (R, ρ) is a Picard–Vessiot extension associated to the system (A) over the field K_∞.*

Proof. For the proof of this theorem we refer to Ch. 4 of [4]. □

Definition 22. We define the ring of *Liouvillian* sequences $\mathscr{L}$ recursively. $\mathscr{L}$ is the smallest subring of $\mathscr{S}_{\mathbb{C}}$ such that

(i) $K_\infty \subset \mathscr{L}$.

(ii) $a \in \mathscr{L}$ implies that $\rho(a) \in \mathscr{L}$.

(iii) $a \in K_\infty$ implies that $b \in \mathscr{L}$ if $b_{n+1} = a_n b_n$ for all but finitely many $n \in \mathbb{N}_{\geq 1}$.

(iv) $a \in \mathscr{L}$ implies that $b \in \mathscr{L}$ if $b_{n+1} = a_n + b_n$ for all but finitely many $n \in \mathbb{N}_{\geq 1}$.

(v) $a \in \mathscr{L}$ implies that $b \in \mathscr{L}$ if there exist a $j \in \mathbb{N}_{\geq 1}$ such that $b_{jn} = a_n$ and $b_n = 0$ if $n \not\equiv 0 \bmod j$. In this case we call b an interlacing of a with zeroes.

Theorem 23. *Suppose $a \in \mathscr{S}_{\mathbb{C}}$. Then the following statements are equivalent:*

(i) $a \in \mathscr{L}$.

(ii) *The sequence a satisfies a linear q-difference equation over K_∞ so that the q-difference Galois group G associated to this equation is solvable.*

Proof. For the proof we refer to [1]. □

A consequence of the proof of the previous theorem is that all the solutions of a second order q-difference equation $\phi^2 y + a\phi y + by = 0$ are Liouvillian if and only if the difference Galois group G is reducible or irreducible and imprimitive. If the q-difference Galois group G contains $Sl(2, \mathbb{C})$, then there are no Liouvillian solutions of the second order q-difference equation $\phi^2 y + a\phi y + by = 0$.

Example. Let $q = 4$ and $q_2 = 2$. Let $a = 6z$ and $b = 2z^2 - 2z$. Consider the q-difference equation $\phi^2 y + a\phi(y) + by = 0$ and the corresponding Riccati equation $\phi(u)u + au + b = 0$. We are in the case that $v_0(b) \leq 2v_0(a)$ and $v_0(b)$ is odd. Therefore, the Riccati equation has not a solution in K_1. Now we have to apply step 4 of the algorithm to find out whether there are solutions in K_2. From now on we will consider a and b as elements in $\mathbb{Q}(z_2)$. The Riccati equation reads $\phi(u)u - 6z_2^2 u + 2z_2^4 - 2z_2^2 = 0$. A calculation shows that $S_0 = \{z_2 + O(z_2^2), -z_2 + O(z_2^2)\}$ and $S_\infty = \{z_2^2 + O(z_2), \frac{1}{2}z_2^2 + O(z_2)\}$. Further we have $S_p = \{1, z_2, z_2 - 1, z_2 + 1, z_2^2, z_2^2 - z_2, z_2^2 + z_2, z_2^2 - 1, z_2^3 - z_2^2, z_2^3 + z_2^2, z_2^3 - z_2, z_2^4 - z_2^2\}$ and $S_t = \{1\}$. We must have $m = v_0(u) + v_0(p) - v_0(t)$ and $m = v_\infty(t) - v_\infty(p) - v_\infty(u)$. Hence $v_0(p) + v_\infty(p) = 1$. Therefore $p = z_2 \pm 1$ and $m = 1$.

Suppose now $u_0 = z_2 + O(z_2^2)$, $u_\infty = z_2^2 + O(z_2)$ and $p = z_2 + 1$; then we find $u_0 t / c z_2^m p = 1/c + O(z_2)$. Hence $c = 1$, and we find $u_\infty t / c z_2^m p = 1 + O(1/z_2)$. Now q^e must be equal to 1. Hence $e = 0$. $R = 1$ satisfies the equation (‡) and one finds that $u_1 = z_2 p = z_2^2 + z_2$ is a solution of the Riccati equation in K_2.

Suppose now $u_0 = -z_2 + O(z_2^2)$, $u_\infty = z_2^2 + O(z_2)$ and $p = z_2 - 1$ then we find $u_0 t / c z_2^m p = 1/c + O(z_2)$. Hence $c = 1$, and we find $u_\infty t / c z_2^m p = 1 + O(1/z_2)$. Now q^e must be equal to 1. Hence $e = 0$. $R = 1$ satisfies the equation (‡) and one finds that $u_2 = z_2 p = z_2^2 - z_2$ is a solution of the Riccati equation in K_2.

A calculation shows that the other combinations of u_0, u_∞ and p do not yield any solution of the Riccati equation in K_2. The difference equation $\phi^2 y + a\phi y + by = 0$ is equivalent to the system

$$\phi y = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} y.$$

This system is already in standard form. Hence,

$$G = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \middle| \, a \in \mathbb{C}^*, \, d \in \mathbb{C}^* \right\}.$$

Two linearly independent elementary solutions of the q-difference equation in the ring $\mathscr{S}_{\mathbb{C}}$ are c and d, where $c_n = \prod_{k=1}^{n-1}(2^{2n} + 2^n)$ and $d_n = \prod_{k=1}^{n-1}(2^{2n} - 2^n)$. The sequence $c \in \mathscr{S}_{\mathbb{C}}$ satisfies the first-order q-difference equation $\phi(y) = u_1 y$ and the sequence $d \in \mathscr{S}_{\mathbb{C}}$ satisfies the first-order q-difference equation $\phi(y) = u_2 y$.

5.2. *Hypergeometric q-difference equations*

In this section we apply the algorithm to some cases of the hypergeometric q-difference equation which is given by

$$\phi^2(y) + \frac{(-4 + q^\alpha + q^\beta)z + 3 - q^{\gamma-1}}{z-1}\phi(y) + \frac{(2 - q^\alpha)(2 - q^\beta)z - 2 + q^{\gamma-1}}{z-1} y = 0.$$

As usual, we will identify this equation with the system (A):

$$\phi y = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix} y,$$

where

$$a = \frac{(-4 + q^\alpha + q^\beta)z + 3 - q^{\gamma-1}}{z-1} \quad \text{and} \quad b = \frac{(2 - q^\alpha)(2 - q^\beta)z - 2 + q^{\gamma-1}}{z-1}.$$

Case 1. We will analyze the case where α, β and γ are rational parameters and q is transcendental. Note that the hypergeometric equation is symmetric in the parameters α and β. This restricts the number of possibilities we have to consider.

Theorem 24. *Let G denote the q-difference Galois group of the hypergeometric q-difference equation.*

(i) *If $\alpha = 0$, $\beta = 0$ and $\gamma = 1$ then*

$$G = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| \, b \in \mathbb{C} \right\}.$$

(ii) *If ($\alpha = 0$ and $\beta = \gamma - 1$) or ($\alpha = \gamma - 1$ and $\beta = 0$) and $\gamma \neq 1$ then*

$$G = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \middle| \, d \in \mathbb{C}^* \right\}.$$

(iii) *If* ($\alpha=0$ *and* $\beta\neq\gamma-1$) *or* ($\alpha\neq\gamma-1$ *and* $\beta=0$) *then*

$$G=\left\{\begin{pmatrix}1 & b\\ 0 & d\end{pmatrix}\middle|\, b\in\mathbb{C},\ d\in\mathbb{C}^*\right\}.$$

(iv) *If* ($\alpha\neq 0$ *and* $\beta=\gamma-1$) *or* ($\alpha=\gamma-1$ *and* $\beta\neq 0$) *then*

$$G=\left\{\begin{pmatrix}a & b\\ 0 & d\end{pmatrix}\middle|\, b\in\mathbb{C},\ a,d\in\mathbb{C}^*\right\}.$$

(v) *If* $\alpha\neq 0$, $\alpha\neq\gamma-1$, $\beta\neq 0$ *and* $\beta\neq\gamma-1$ *then* $G=Gl(2,\mathbb{C})$.

Proof. Consider the rather special cases that ($\alpha=0$ and $\beta=\gamma-1$) or ($\alpha=\gamma-1$ and $\beta=0$). In these cases the hypergeometric q-difference equation simplifies to the equation $\phi^2(y)-(3-q^{\gamma-1})\phi(y)+(2-q^{\gamma-1})y=0$.

If moreover $\gamma=1$, then we get the equation $\phi^2(y)-2\phi(y)+y=0$. In this case $u=1$ is the only solution of the corresponding Riccati equation. Hence, the q-difference Galois group is reducible but not completely reducible. Let

$$T=\begin{pmatrix}0 & 1\\ -1 & 1\end{pmatrix}\quad\text{and}\quad B=\phi(T)AT^{-1}=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}.$$

Obviously, the equivalent system $(B):\phi(y)=By$ is in standard form. Hence, the q-difference Galois group G is equal to the group

$$\left\{\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}\middle|\, b\in\mathbb{C}\right\}.$$

Suppose now that $\gamma\neq 0$. Clearly, $u=1$ and $u=2-q^{\gamma-1}$ are the only solutions of the corresponding Riccati equation. Hence, the q-difference Galois group G is completely irreducible if $\gamma\neq 0$. In that case the q-difference equation is equivalent to the system of difference equations

$$\phi y=\begin{pmatrix}1 & 0\\ 0 & 2-q^{\gamma-1}\end{pmatrix}y.$$

This system is in standard form. The q-difference Galois group G is equal to the group

$$\left\{\begin{pmatrix}1 & 0\\ 0 & d\end{pmatrix}\middle|\, d\in\mathbb{C}^*\right\},$$

because q is transcendental and γ is rational.

From now on we will exclude the cases that ($\alpha=0$ and $\beta=\gamma-1$) or ($\alpha=\gamma-1$ and $\beta=0$).

Consider the Riccati equation $(\dagger):\phi(u)u+au+b=0$. A simple calculation shows that $S_0=\{1+O(z),2-q^{\gamma-1}+O(z)\}$ and $S_\infty=\{2-q^\alpha+O(1/z),2-q^\beta+O(1/z)\}$.

Further we have

$$S_p = \left\{1,\ z - \frac{2 - q^{\gamma-1}}{(2-q^{\alpha})(2-q^{\beta})}\right\}$$

and $S_t = \{1, z - q\}$. For all choices $u_0 \in S_0$, $p \in S_p$ and $t \in S_t$ we find $m = 0$. For all $u_\infty \in S_\infty$ we have $v_\infty(u_\infty) = 0$. Hence, we must have $\deg_z(p) = \deg_z(t)$. Now we have to consider four different possibilities.

The first combination is $u_0 = 1 + \mathrm{O}(z)$, $p = 1$, $t = 1$ and $u_\infty = 2 - q^{\alpha} + \mathrm{O}(1/z)$. We must have $u_0 t/cp = 1 + \mathrm{O}(z)$. Hence, $c = 1$. And we compute $u_\infty t/cp = 2 - q^{\alpha} + \mathrm{O}(1/z)$. So $d = 2 - q^{\alpha}$. We have d is of the form q^e with $e \in \mathbb{Z}_{\geq 0}$ if and only if $\alpha = 0$. In that case $e = 0$ and $R = 1$ turns out to be a solution of the equation (‡). Therefore, $u = c(\phi(R)p/Rt) = 1$ is a solution of the Riccati equation if $\alpha = 0$.

The second combination is $u_0 = 2 - q^{\gamma-1} + \mathrm{O}(z)$, $p = 1$, $t = 1$ and $u_\infty = 2 - q^{a} + \mathrm{O}(1/z)$. We must have $u_0 t/cp = 1 + \mathrm{O}(z)$. Hence $c = 2 - q^{\gamma-1}$. And we compute $u_\infty t/cp = (2 - q^{\alpha})/(2 - q^{\gamma-1}) + \mathrm{O}(1/z)$. So $d = (2 - q^{\alpha})(2 - q^{\gamma-1})$. We have d is of the form q^e with $e \in \mathbb{Z}_{\geq 0}$ if and only if $\alpha = \gamma - 1$. In that case $e = 0$ and $R = 1$ turns out to be a solution of the equation (‡). Therefore, $u = c(\phi(R)p/Rt) = 2 - q^{\gamma-1}$ is a solution of the Riccati equation if $\alpha = \gamma - 1$.

The third combination is $u_0 = 1 + \mathrm{O}(z)$, $p = z - (2 - q^{\gamma-1})/(2 - q^{\alpha})(2 - q^{\beta})$, $t = z - q$ and $u_\infty = 2 - q^{\alpha} + \mathrm{O}(1/z)$. We must have $u_0 t/cp = 1 + \mathrm{O}(z)$. Hence, $c = (2-q^{\gamma-1})/q(2 - q^{\alpha})(2 - q^{\beta})$, and we compute $u_\infty t/cp = q(2 - q^{\alpha})^2(2 - q^{\beta})/(2 - q^{\gamma-1}) + \mathrm{O}(1/z)$. So $d = q(2 - q^{\alpha})^2(2 - q^{\beta})/(2 - q^{\gamma-1})$. Now d is not of the form q^e with $e \in \mathbb{Z}_{\geq 0}$, unless we have $\alpha = 0$ and $\beta = \gamma - 1$. But we have excluded this situation. Hence, this combination does not yield a rational solution of the Riccati equation.

The fourth combination is $u_0 = 2 - q^{\gamma-1} + \mathrm{O}(z)$, $p = z - (2 - q^{\gamma-1})/(2 - q^{\alpha})(2 - q^{\beta})$, $t = z - q$ and $u_\infty = 2 - q^{\alpha} + \mathrm{O}(1/z)$. We must have $u_0 t/cp = 1 + \mathrm{O}(z)$. Hence, $c = 1/q(2 - q^{\alpha})(2 - q^{\beta})$, and we compute $u_\infty t/cp = q(2 - q^{\alpha})^2(2 - q^{\beta}) + \mathrm{O}(1/z)$. So $d = q(2 - q^{\alpha})^2(2 - q^{\beta})$. Now d is of the form q^e with $e \in \mathbb{Z}_{\geq 0}$ if and only if $\alpha = \beta = 0$. Then $e = 1$. We substitute $R = z + r$ in (‡) (see Section 4.1). We get the equation $(1/q^2)(q^2 z + r)\phi(p)p(z - 1) + (1/q)(qz + r)\phi(t)p(-2z + 3 - q^{\gamma-1}) + (z + r)\phi(t)t(z - (2 - q^{\gamma-1}) = 0$. After simplifying this equation we get two linear equations for r that contradict each other. Hence, this combination does not yield any solution of the Riccati equation.

We found that there is exactly one solution of the Riccati equation in K if $\alpha = 0$ and $\beta \neq \gamma - 1$ or $\alpha = \gamma - 1$ and $\beta \neq 0$ or $\beta = 0$ and $\alpha \neq \gamma - 1$ or $\beta = \gamma - 1$ and $\alpha \neq 0$. Applying step 4 of the algorithm for solving the Riccati equation we find that all the solutions of the Riccati equation in K_∞ are already in K. This computation is similar to the computations above.

Suppose that $\alpha = 0$ and $\beta \neq \gamma - 1$. Let

$$T = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}.$$

Then we get

$$B=\phi(T)AT^{-1}=\begin{pmatrix} 1 & \dfrac{(2-q^{\beta})z-(2-q^{\gamma-1})}{z-1} \\ 0 & \dfrac{(2-q^{\beta})z-(2-q^{\gamma-1})}{z-1} \end{pmatrix}.$$

System (B) is equivalent to system (A). Moreover, system (B) is in standard form because q is transcendental and $\beta \neq \gamma-1$. The differential Galois group

$$G=\left\{\begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix} \,\middle|\, b\in\mathbb{C},\ d\in\mathbb{C}^*\right\}.$$

Suppose that $\alpha=\gamma-1$ and $\beta\neq 0$. Let

$$T=\begin{pmatrix} -1+2^{\gamma-1} & 1 \\ -2+2^{\gamma-1} & 1 \end{pmatrix}.$$

Then we get

$$B=\phi(T)AT^{-1}=\begin{pmatrix} 2-q^{\gamma-1} & \dfrac{(1+q^{\gamma-1}-q^{\beta})z-q^{\gamma-1}}{z-1} \\ 0 & \dfrac{(2-q^{\beta})z-1}{z-1} \end{pmatrix}.$$

System (B) is equivalent to system (A). Moreover, system (B) is in standard form because q is transcendental and $\beta\neq 0$. The q-difference Galois group

$$G=\left\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \,\middle|\, b\in\mathbb{C},\ a,d\in\mathbb{C}^*\right\}.$$

If $\alpha\neq 0$, $\alpha\neq\gamma-1$, $\beta\neq 0$ and $\beta\neq\gamma-1$ then the q-difference Galois group is irreducible. By using the algorithms of Sections 4.3 and 4.4 one can show that in this case the q-difference Galois group G is the group $Gl(2,\mathbb{C})$. □

It is possible to give two linearly independent Liouvillian solution for the hypergeometric q-difference equations in all those cases where the q-difference Galois group G is not the group $Gl(2,\mathbb{C})$.

Case 2. If q is not transcendental then the situation can be different. For instance, let $\alpha=\beta=\gamma=1$. Then the equation simplifies to

$$\phi^2(y)+\frac{(-4+2q)z+2}{z-1}\phi(y)+\frac{(2-q)^2z-1}{z-1}y=0.$$

If q is transcendental then the q-difference Galois group G is the group $Gl(2,\mathbb{C})$. But if $q=-\frac{1}{2}\pm\frac{1}{2}\sqrt{-7}$, that is q is a root of the polynomial x^2+x+2, then the difference Galois group G turns out to be reducible. From now on we will assume that $q=-\frac{1}{2}\pm\frac{1}{2}\sqrt{-7}$.

Consider the corresponding Riccati equation $(\dagger)$: $\phi(u)u + au + b = 0$, where $a = ((-4+2q)z+2)/(z-1)$ and $b = ((2-q)^2 z - 1)/(z-1)$. We find $S_0 = \{1 + O(z)\}$ and $S_\infty = \{2 - q + O(1/z)\}$. Further we have $S_p = \{1, z - 1/(2-q)^2\}$ and $S_t = \{1, z - q\}$. We must have $\deg_z(p) = \deg_z(t)$. Hence, we have to consider two combinations.

The first combination is $u_0 = 1 + O(z)$, $p = 1$, $t = 1$, and $u_\infty = 2 - q + O(1/z)$. We must have $u_0 t/cp = 1 + O(z)$. Hence $c = 1$. We compute $u_\infty t/cp = 2 - q + O(1/z)$. So $d = 2 - q$. We have $2 - q = q^3$. Hence $e = 3$. We substitute $R = z^3 + r_2 z^2 + r_1 z + r_0$ in the equation $(\ddagger)$, and we find the solution

$$R = z^3 - \frac{3q+1}{16}z^2 + \frac{5q+7}{64}z + \frac{5q+7}{64} = \left(z - \frac{1-q}{4}\right)\left(z^2 + \frac{3-7q}{16}z - \frac{1-3q}{16}\right).$$

Now $u = \phi(R)/R$ is a solution of the Riccati equation.

The second combination is $u_0 = 1 + O(z)$, $p = z - 1/(2-q)^2$, $t = z - q$, and $u_\infty = 2 - q + O(1/z)$. We must have $u_0 t/cp = 1 + O(z)$. Hence, $c = 1/q(2-q)^2$. We compute $u_\infty t/cp = q(2-q)^3 + O(1/z)$. So $d = q(2-q)^3 = q^{10}$. Hence $e = 10$. We substitute $R = z^{10} + r_9 z^9 + \cdots + r_1 z + r_0$ in $(\ddagger)$. After simplifying we get eleven linear equations for the ten variables r_i. There are no solutions. We conclude that the second combination does not yield any solution of the Riccati equation.

We have $u = \phi(R)/R$ with

$$R = \left(z - \frac{1-q}{4}\right)\left(z^2 + \frac{3-7q}{16}z - \frac{1-3q}{16}\right)$$

is the only solution of the Riccati equation in K_∞. Let

$$T = \begin{pmatrix} 1-u & 1 \\ -u & 1 \end{pmatrix}.$$

Then we get

$$B = \phi(T)(\ )T^{-1} = \begin{pmatrix} u & 1 - u + b/u \\ 0 & b/u \end{pmatrix}.$$

System (B) is not yet in standard form. Note that $u = \phi(R)/R$ and $b = \phi(s)/s$, where $s = (q^5 z - 1)(q^4 z - 1)\cdots(qz-1)(z-1)$. Let

$$S = \begin{pmatrix} \frac{1}{R} & 0 \\ 0 & \frac{R}{s} \end{pmatrix}.$$

Then we get

$$C = \phi(S)BS^{-1} = \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \quad \text{where } f = \frac{(u - u^2 + b)s}{\phi(R)^2}.$$

System (C) is in standard form. Therefore, the q-difference Galois group G is equal to the group

$$\left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| \, b \in \mathbb{C} \right\}.$$

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